2C2: P.D.E., Handout 8

Niels R. Walet, December 10, 2002
Niels.Walet@umist.ac.uk, **http://walet.phy.umist.ac.uk/2C1/**

3D problems and Legendre polynomials

1 Legendre's equation

A basic equation, obtained when separating variables in 3D is,

$$[\sin\theta T']' + \lambda T \sin\theta = 0.$$

This equation is called Legendre's equation after changing variables to $x = \cos\theta$. The variable θ runs from 0 to π and thus $\sin\theta > 0$. We have

$$\sin\theta = \sqrt{1-x^2}.$$

After this substitution, with $y(x) = T(\theta) = T(\arccos x)$, using $d/d\theta = -\sqrt{1-x^2}d/dx$) we get

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + \lambda y = 0.$$

This equation is self-adjoint. $x = 0$ is a regular (not singular) point – but the equation *is* singular at $x = \pm 1$. Near $x = 0$ we can solve it as a Taylor series,

$$y(x) = \sum_{j=0} a_j x^j.$$

We find the equation

$$\sum_{j=0}^{\infty} j(j-1)a_j x^{j-2} - \sum_{j=0}^{\infty} j(j-1)a_j x^j - 2\sum_{j=0}^{\infty} j a_j x^j + \lambda \sum_{j=0}^{\infty} a_j x^j = 0$$

After introducing the new variable $i = j - 2$, we have

$$\sum_{j=0}^{\infty}(i+1)(i+1)a_{i+2}x^i - \sum_{j=0}^{\infty}[j(j+1)-\lambda]a_j x^j = 0.$$

Collecting the terms of the order x^k, we find the recurrence relation

$$a_{k+2} = \frac{k(k+1)-\lambda}{(k+1)(k+2)}a_k.$$

If $\lambda = n(n+1)$ this series terminates – actually those are the only acceptable solutions, any one where λ takes a different value actually diverges at $x = +1$ or $x = -1$, not acceptable for a physical quantity.

One conventionally defines

$$a_n = \frac{(2n)!}{n!^2 2^n}.$$

With this definition we obtain

$$\begin{array}{lll} P_0 = 1, & & P_1 = x, \\ P_2 = \frac{3}{2}x^2 - \frac{1}{2}, & & P_3 = \frac{1}{2}(5x^3 - 3x), \\ P_4 = \frac{1}{8}(35x^4 - 30x^2 + 3), & & P_5 = \frac{1}{8}(63x^5 - 70x^3 + 15x). \end{array}$$

A graph of these polynomials can be found in figure 1.

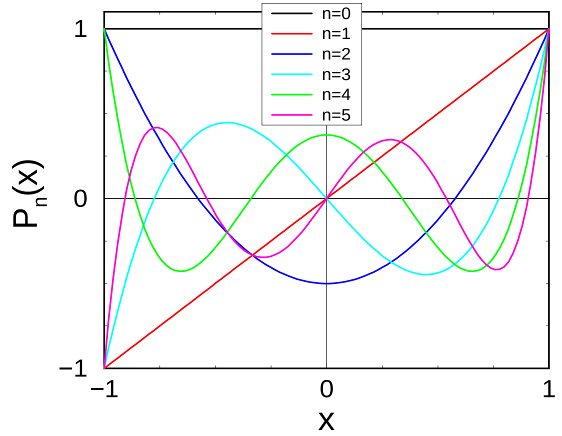

Figure 1: The first few Legendre polynomials $P_n(x)$.

2 Properties of Legendre polynomials

2.1 Generating function

Let $F(x,t)$ be a function of x and t that can be expressed as a Taylor's series in t, $\sum_n c_n(x)t^n$. The function F is called a generating function of the functions $c_n(x)$.

Example 1:

$F(x,t)=\frac{1}{1-xt}$ is a generating function of the polynomials x^n since

$$\frac{1}{1-xt}=\sum_{n=0}^{\infty}x^n t^n \quad (|xt|<1).$$

Example 2:

$F(x,t)=\exp\left(\frac{x}{2}(t-1/t)\right)$ is the generating function of the Bessel functions,

$$F(x,t)=\exp(x(t-1/t))=\sum_{n=0}^{\infty}J_n(x)t^n.$$

Example 3:

(The case of most interest here)

$$F(x,t)=\frac{1}{\sqrt{1-2xt+t^2}}=\sum_{n=0}^{\infty}P_n(x)t^n.$$

2.2 Rodrigue's Formula

$$P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n.$$

2.3 A table of properties

1. $P_n(x)$ is even or odd if n is even or odd.
2. $P_n(1)=1$.
3. $P_n(-1)=(-1)^n$.
4. $(2n+1)P_n(x)=P'_{n+1}(x)-P'_{n-1}(x)$.
5. $(2n+1)xP_n(x)=(n+1)P_{n+1}(x)+nP_{n-1}(x)$.
6. $\int_{-1}^{x}P_n(x)dx=\frac{1}{2n+1}\left[P_{n+1}(x)-P_{n-1}(x)\right]$.

Let us start from the simple formula

$$(x^2-1)\frac{d}{dx}(x^2-1)^n-2nx(x^2-1)^n=0,$$

This is differentiated $n+1$ times,

$$\begin{aligned}
&\frac{d^{n+1}}{dx^{n+1}}\left[(x^2-1)\frac{d}{dx}(x^2-1)^n-2nx(x^2-1)^n\right]\\
&= n(n+1)\frac{d^n}{dx^n}(x^2-1)^n+2(n+1)x\frac{d^{n+1}}{dx^{n+1}}(x^2-1)^n+(x^2-1)\frac{d^{n+2}}{dx^{n+2}}(x^2-1)^n\\
&\quad -2n(n+1)\frac{d^n}{dx^n}(x^2-1)^n-2nx\frac{d^{n+1}}{dx^{n+1}}(x^2-1)^n\\
&= -\left[\frac{d}{dx}(1-x^2)\frac{d}{dx}\left\{\frac{d^n}{dx^n}(x^2-1)^n\right\}+n(n+1)\left\{\frac{d^n}{dx^n}(x^2-1)^n\right\}\right]=0.
\end{aligned}$$

We have thus proven that $\frac{d^n}{dx^n}(x^2-1)^n$ satisfies Legendre's equation. The normalization follows from the evaluation of the highest coefficient,

$$\frac{d^n}{dx^n}x^{2n}=\frac{2n!}{n!}x^n.$$

Let's use the generating function to prove some of the other properties: 2.:

$$F(1,t)=\frac{1}{1-t}=\sum_n t^n$$

has all coefficients one, so $P_n(1)=1$. Similarly for 3.:

$$F(-1,t)=\frac{1}{1+t}=\sum_n(-1)^n t^n.$$

Property 5. can be found by differentiating the generating function with respect to t:

$$\begin{aligned}
\frac{d}{dt}\frac{1}{\sqrt{1-2tx+t^2}} &= \frac{d}{dt}\sum_{n=0}^{\infty} t^n P_n(x) \\
\frac{x-t}{(1-2tx+t^2)^{3/2}} &= \sum_{n=0} n t^{n-1} P_n(x) \\
\frac{x-t}{1-2xt+t^2}\sum_{n=0}^{\infty} t^n P_n(x) &= \sum_{n=0} n t^{n-1} P_n(x) \\
\sum_{n=0}^{\infty} t^n x P_n(x) - \sum_{n=0}^{\infty} t^{n+1} P_n(x) &= \sum_{n=0}^{\infty} n t^{n-1} P_n(x) - 2\sum_{n=0}^{\infty} n t^n x P_n(x) + \sum_{n=0}^{\infty} n t^{n+1} P_n(x) \\
\sum_{n=0}^{\infty} t^n (2n+1) x P_n(x) &= \sum_{n=0}^{\infty} (n+1) t^n P_{n+1}(x) + \sum_{n=0}^{\infty} n t^n P_{n-1}(x)
\end{aligned}$$

Equating terms with identical powers of t we find

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x).$$

Proofs can be found for the other properties using similar methods.

3 Fourier-Legendre series

Since Legendre's equation is self-adjoint, $P_n(x)$ forms an orthogonal set of functions. To decompose functions as series in Legendre polynomials we need the integrals

$$\int_{-1}^{1} P_n^2(x)dx = \frac{2}{2n+1}$$

We use relation 5. twice we obtain recurrence relation

$$\begin{aligned}
\int_{-1}^{1} P_n^2(x)dx &= \int_{-1}^{1} P_n(x)\frac{(2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x)}{n}dx \\
&= \frac{(2n-1)}{n}\int_{-1}^{1} xP_n(x)P_{n-1}(x)dx \\
&= \frac{(2n-1)}{n}\int_{-1}^{1} \frac{(n+1)P_{n+1}(x) + nP_{n-1}(x)}{2n+1}P_{n-1}(x)dx \\
&= \frac{(2n-1)}{2n+1}\int_{-1}^{1} P_{n-1}^2(x)dx,
\end{aligned}$$

and fix this number for $n = 0$,

$$\int_{-1}^{1} P_0^2(x)dx = 2.$$

So we can now develop any function on $[-1, 1]$ in a Fourier-Legendre series

$$f(x) = \sum_n A_n P_n(x), \qquad A_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)dx$$

Example 4:

Find the Fourier-Legendre series for

$$f(x) = \begin{cases} 0 & -1 < x < 0 \\ 1 & 0 < x < 1 \end{cases}.$$

Solution:

We find

$$\begin{aligned}
A_0 &= \tfrac{1}{2}\int_0^1 P_0(x)dx = \tfrac{1}{2}, & A_1 &= \tfrac{3}{2}\int_0^1 P_1(x)dx = \tfrac{1}{4}, \\
A_2 &= \tfrac{5}{2}\int_0^1 P_2(x)dx = 0, & A_3 &= \tfrac{7}{2}\int_0^1 P_3(x)dx = -\tfrac{7}{16}.
\end{aligned}$$

All other coefficients for even n are zero, for odd n they can be evaluated explicitly.

4 Modeling the eye

Let me model the temperature in a simple model of the eye, where the eye is a sphere, and the eyelids are circular. We can assume that the temperature does only depend on r, θ and *not* on ϕ. We assume that the part of the eye in contact with air is at a temperature of 20° C, and the part in contact with the body is at 36° C. If we look for the steady state temperature it is described by Laplace's equation,

$$\nabla^2 u(r, \theta) = 0.$$

Expressing the Laplacian ∇^2 in spherical coordinates we find

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) = 0.$$

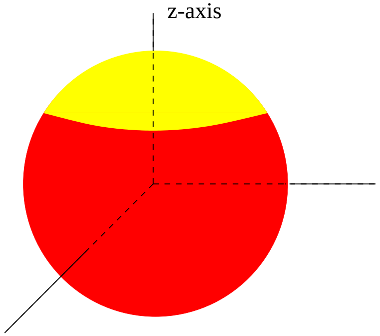

Figure 2: The temperature in a simple model of the eye

Once again we solve the equation by separation of variables,

$$u(r,\theta) = R(r)T(\theta).$$

After this substitution we realize that

$$\frac{[r^2R']'}{R} = -\frac{[\sin\theta T']'}{T\sin\theta} = \lambda.$$

With the functions $P_n(\cos\theta)$ as the solution to the angular equation, we find that the solutions to the radial equation are

$$R = Ar^n + Br^{-n-1}.$$

The singular part is not acceptable, so the solution takes the form

$$u(r,\theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta)$$

Impose the boundary condition that the temperature is 20° C in an opening angle of 45°, and 36° elsewhere. This leads to the equation

$$\sum_{n=0}^{\infty} A_n c^n P_n(\cos\theta) = \begin{cases} 20 & 0 < \theta < \pi/4 \\ 36 & \pi/4 < \theta < \pi \end{cases}$$

This leads to the integral, after once again changing to $x = \cos\theta$,

$$A_n = \frac{2n+1}{2}\left[\int_{-1}^{1} 36P_n(x)dx - \int_{\frac{1}{2}\sqrt{2}}^{1} 16P_n(x)dx\right].$$

These integrals can easily be evaluated, and a sketch for the temperature can be found in figure 3.

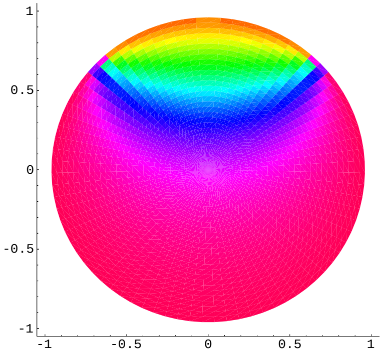

Figure 3: A cross-section of the temperature in the eye. We have summed over the first 40 Legendre polynomials.